SYMMETRIES $\Longrightarrow$ INTERACTIONS
Phase Invariance (Symmetry) in Quantum Mechanics

QM STATE: COMPLEX SCHRÖDINGER WAVE FUNCTION $\psi(x)$

OBSERVABLES

$$\langle O \rangle = \int d^n x \psi^* O \psi$$

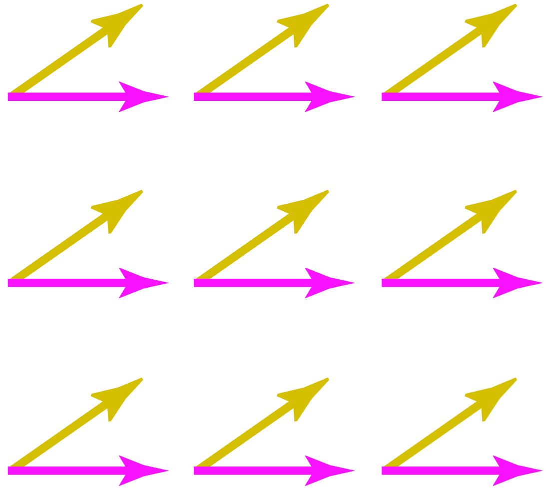
ARE UNCHANGED

UNDER A GLOBAL PHASE ROTATION

$$\psi(x) \to e^{i\theta}\psi(x)$$
$$\psi^*(x) \to e^{-i\theta}\psi^*(x)$$

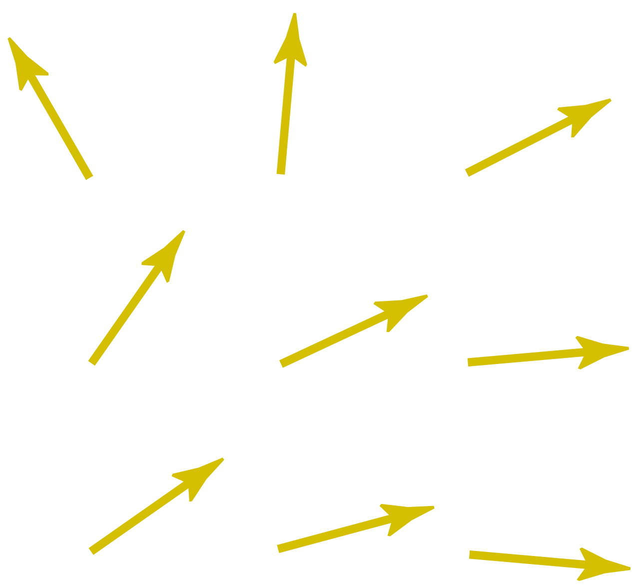
- Absolute phase of the wave function cannot be measured (is a matter of convention).
- Relative phases (interference experiments) are unaffected by a global phase rotation.

GLOBAL ROTATION — SAME EVERYWHERE



MIGHT WE CHOOSE ONE PHASE CONVENTION IN GENEVA AND ANOTHER IN BATAVIA?

A DIFFERENT CONVENTION AT EACH POINT?



$$\psi(x) \to e^{iq\alpha(x)}\psi(x)$$

THERE IS A PRICE.

Some variables (e.g., momentum) and the Schrödinger equation itself contain derivatives. Under the transformation

$$\psi(x) \to e^{iq\alpha(x)}\psi(x)$$

the gradient of the wave function transforms as

$$\nabla\psi(x) \to e^{iq\alpha(x)}[\nabla\psi(x) + iq(\nabla\alpha(x))\psi(x)]$$

The $\nabla\alpha(x)$ term spoils local phase invariance.

TO RESTORE LOCAL PHASE INVARIANCE ...

Modify the equations of motion and observables.

$$\boxed{\text{Replace } \nabla \text{ by } \nabla + iq\vec{A}}$$

"Gauge-covariant derivative"

If the vector potential $\vec{A}$ transforms under local phase rotations as

$$\vec{A}(x) \to \vec{A}'(x) \equiv \vec{A}(x) - \nabla\alpha(x),$$

then $(\nabla + iq\vec{A})\psi \to e^{iq\alpha(x)}(\nabla + iq\vec{A})\psi$ and $\psi^*(\nabla + iq\vec{A})\psi$ is invariant under local rotations.

NOTE . . .

- $\vec{A}(x) \to \vec{A}'(x) \equiv \vec{A}(x) - \nabla\alpha(x)$ has the form of a gauge transformation in electrodynamics.
- The replacement $\nabla \to (\nabla + iq\vec{A})$ corresponds to $\vec{p} \to \vec{p} - q\vec{A}$

FORM OF INTERACTION IS DEDUCED
FROM LOCAL PHASE INVARIANCE

$\Longrightarrow$ MAXWELL'S EQUATIONS

DERIVED

FROM A SYMMETRY PRINCIPLE

QED is the gauge theory based on
$U(1)$ phase symmetry

GENERAL PROCEDURE

- Recognize a symmetry of Nature.
- Build it into the laws of physics.
  (Connection with conservation laws)
- Impose symmetry in stricter (local) form.

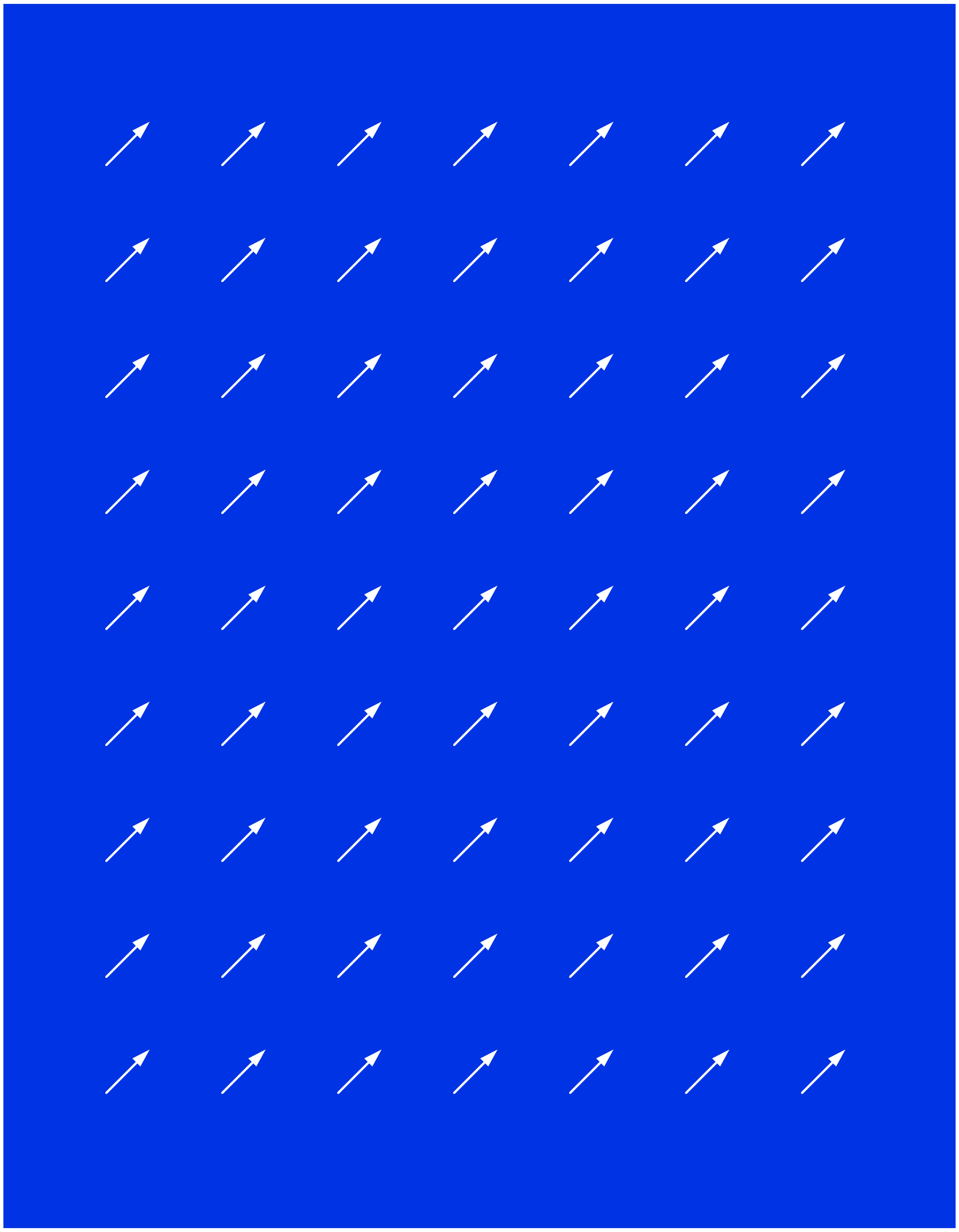
$\Longrightarrow$ INTERACTIONS

- Massless vector fields (gauge fields)
- Minimal coupling to the conserved current
- Interactions among the gauge fields, if symmetry is non-Abelian

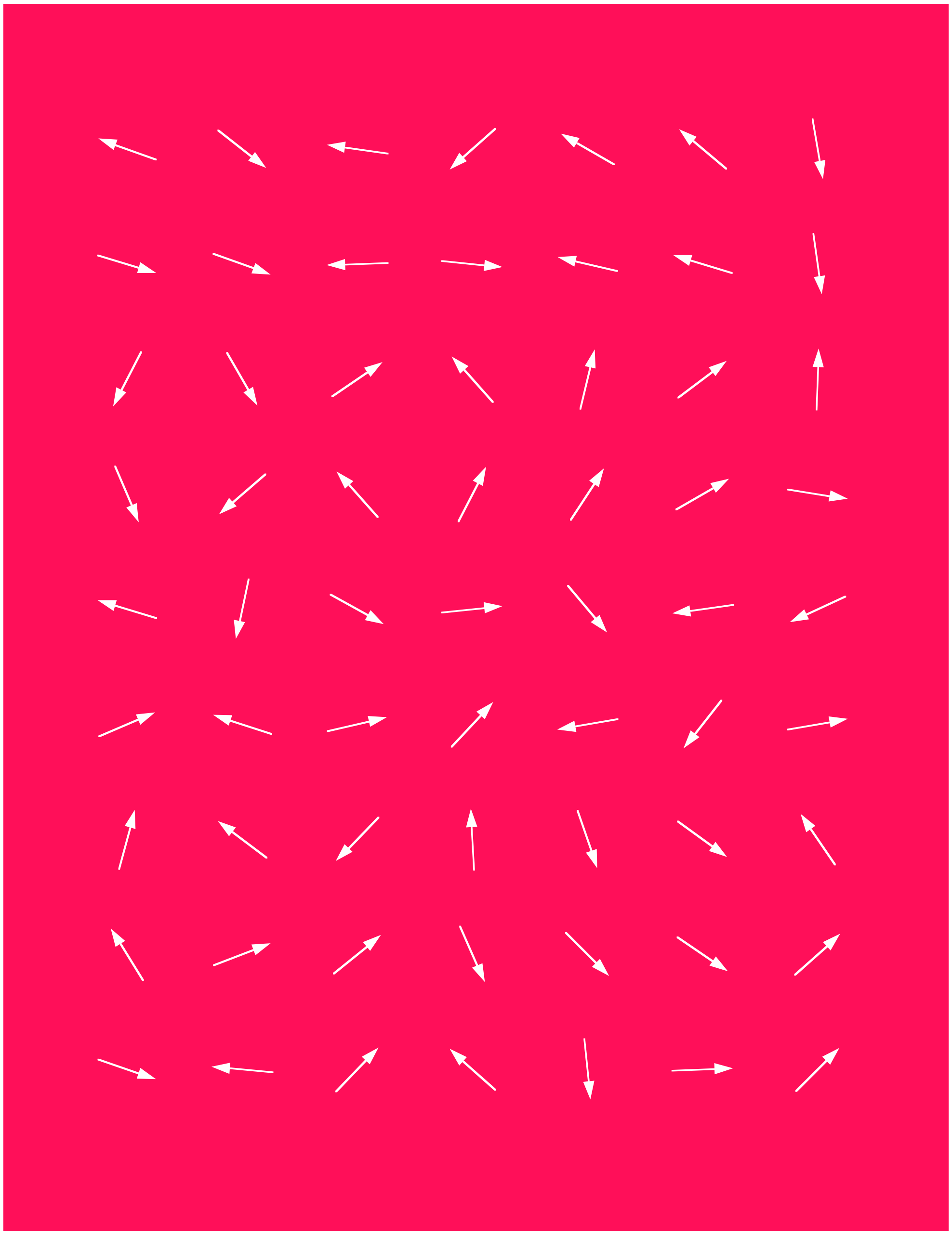
Posed as a problem in mathematics, construction of a gauge theory is always possible (at the level of a classical $\mathcal{L}$; consistent quantum theory may require additional vigilance).

Formalism is no guarantee that the gauge symmetry was chosen wisely.

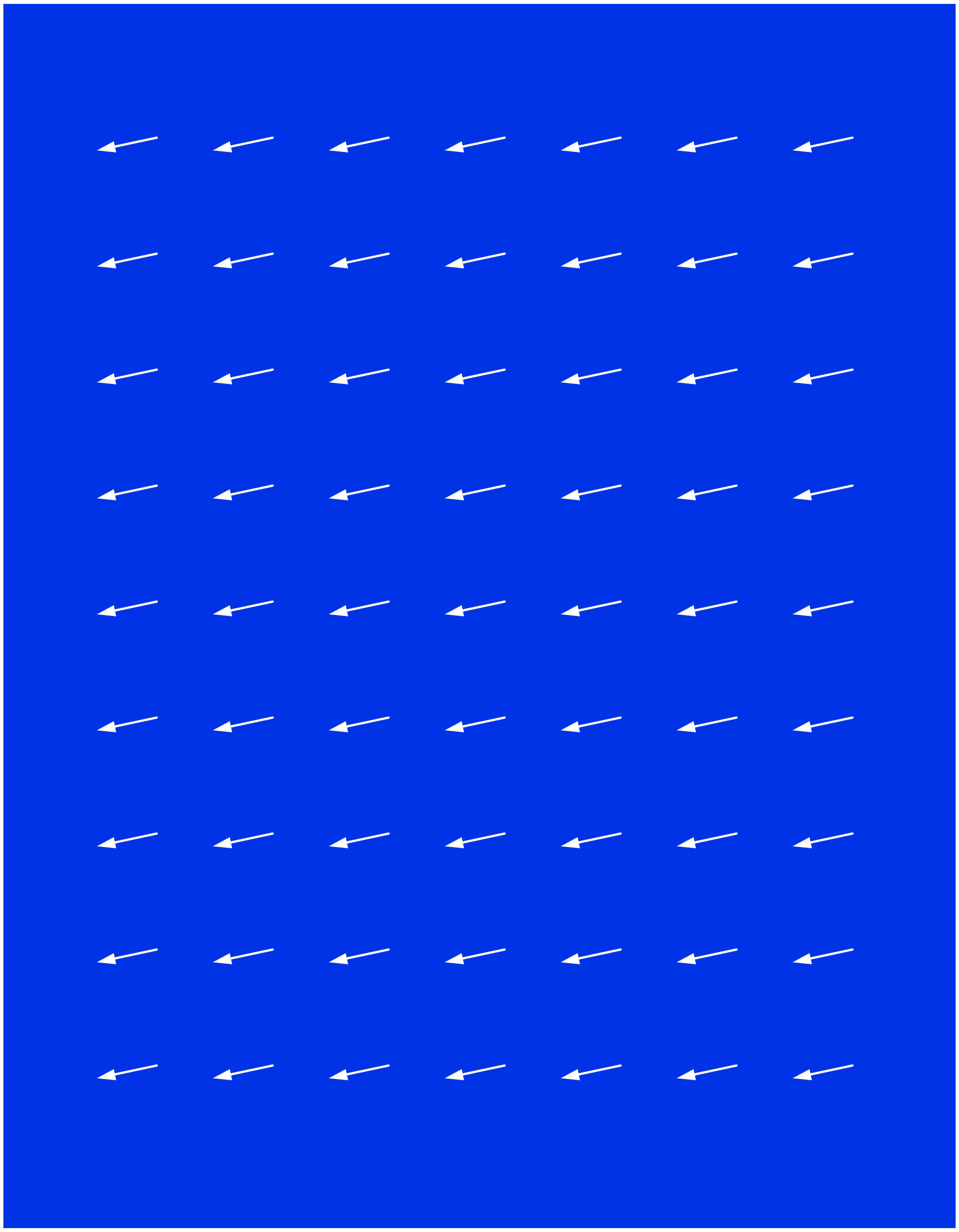
The Crystal World



The Crystal World



The Crystal World



The Perfect World

Recall 2 miracles of superconductivity:

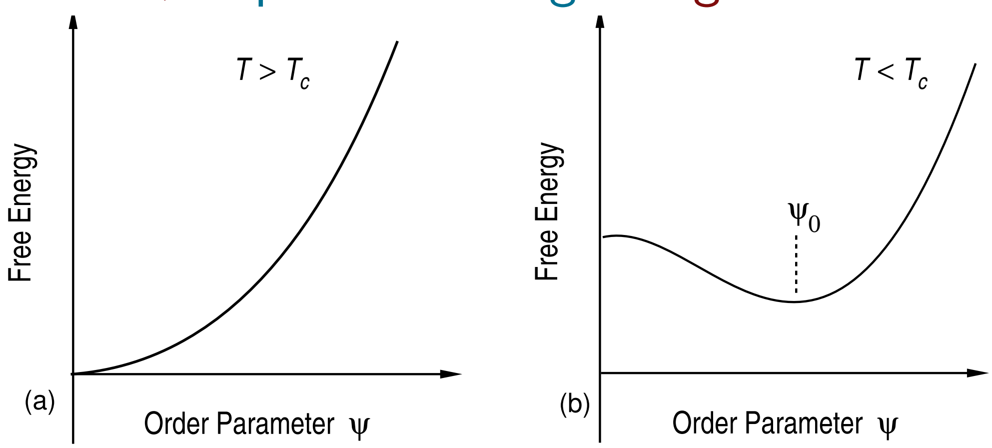
▷ No resistance

▷ Meissner effect (exclusion of $\mathbf{B}$)

Ginzburg–Landau Phenomenology
(not a theory from first principles)

normal, resistive charge carriers . . .

. . . + superconducting charge carriers



$\mathbf{B} = 0$:

$$G_{\text{super}}(0) = G_{\text{normal}}(0) + \alpha\,|\psi|^2 + \beta\,|\psi|^4$$

$$T > T_c: \quad \alpha > 0 \quad \langle|\psi|^2\rangle_0 = 0$$

$$T < T_c: \quad \alpha < 0 \quad \langle|\psi|^2\rangle_0 \neq 0$$

NONZERO MAGNETIC FIELD

$$G_{\text{super}}(\mathbf{B}) = G_{\text{super}}(0) + \frac{\mathbf{B}^2}{8\pi} + \frac{1}{2m^*}\left|-i\hbar\nabla\psi - \frac{e^*}{c}\mathbf{A}\psi\right|^2$$

$$\left.\begin{array}{c} e^* = -2 \\ m^* \end{array}\right\} \text{of superconducting carriers}$$

Weak, slowly varying field

$$\psi \approx \psi_0 \neq 0,\ \nabla\psi \approx 0$$

Variational analysis $\Longrightarrow$

$$\boxed{\nabla^2\mathbf{A} - \frac{4\pi e^*}{m^* c^2}\left|\psi_0\right|^2\mathbf{A} = 0}$$

wave equation of a *massive photon*

Photon— *gauge boson* — acquires mass
within superconductor

origin of Meissner effect

Formulate electroweak theory

three crucial clues from experiment:

▷ Left-handed weak-isospin doublets,

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \qquad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L \qquad \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L$$

and

$$\begin{pmatrix} u \\ d' \end{pmatrix}_L \qquad \begin{pmatrix} c \\ s' \end{pmatrix}_L \qquad \begin{pmatrix} t \\ b' \end{pmatrix}_L ;$$

▷ Universal strength of the (charged-current) weak interactions;

▷ Idealization that neutrinos are massless.

First two clues suggest $SU(2)_L$ gauge symmetry

A theory of leptons

$$\mathsf{L} = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \qquad \mathsf{R} \equiv e_R$$

weak hypercharges $Y_L = -1$, $Y_R = -2$

Gell-Mann–Nishijima connection, $Q = I_3 + \frac{1}{2}Y$

$SU(2)_L \otimes U(1)_Y$ gauge group $\Rightarrow$ gauge fields:

⋆ weak isovector $\vec{b}_\mu$, coupling g

⋆ weak isoscalar $\mathcal{A}_\mu$, coupling $g'/2$

Field-strength tensors

$$F^\ell_{\mu\nu} = \partial_\nu b^\ell_\mu - \partial_\mu b^\ell_\nu + g\varepsilon_{jk\ell} b^j_\mu b^k_\nu \; , SU(2)_L$$

and

$$f_{\mu\nu} = \partial_\nu \mathcal{A}_\mu - \partial_\mu \mathcal{A}_\nu \; , U(1)_Y$$

Interaction Lagrangian

$$\mathcal{L} = \mathcal{L}_{\rm gauge} + \mathcal{L}_{\rm leptons} \;,$$

with

$$\mathcal{L}_{\rm gauge} = -\tfrac{1}{4}F^{\ell}_{\mu\nu}F^{\ell\mu\nu} - \tfrac{1}{4}f_{\mu\nu}f^{\mu\nu},$$

and

$$\begin{aligned}\mathcal{L}_{\rm leptons} &= \overline{\mathsf{R}}\, i\gamma^{\mu}\left(\partial_{\mu} + i\frac{g'}{2}\mathcal{A}_{\mu}Y\right)\mathsf{R} \\ &+ \overline{\mathsf{L}}\, i\gamma^{\mu}\left(\partial_{\mu} + i\frac{g'}{2}\mathcal{A}_{\mu}Y + i\frac{g}{2}\vec{\tau}\cdot\vec{b}_{\mu}\right)\mathsf{L}.\end{aligned}$$

Electron mass term

$$\mathcal{L}_e = -m_e(\bar{e}_{\mathsf{R}}e_{\mathsf{L}} + \bar{e}_{\mathsf{L}}e_{\mathsf{R}}) = -m_e\bar{e}e$$

would violate local gauge invariance Theory has four massless gauge bosons

$$\mathcal{A}_{\mu} \quad b^1_{\mu} \quad b^2_{\mu} \quad b^3_{\mu}$$

Nature has but one (γ)

Hiding EW Symmetry

Higgs mechanism: relativistic generalization of Ginzburg-Landau superconducting phase transition

▷ Introduce a complex doublet of scalar fields

$$\phi \equiv \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \quad Y_\phi = +1$$

▷ Add to $\mathcal{L}$ (gauge-invariant) terms for interaction and propagation of the scalars,

$$\mathcal{L}_{\mathrm{scalar}} = (\mathcal{D}^\mu \phi)^\dagger (\mathcal{D}_\mu \phi) - V(\phi^\dagger \phi),$$

where $\mathcal{D}_\mu = \partial_\mu + i\frac{g'}{2}\mathcal{A}_\mu Y + i\frac{g}{2}\vec{\tau} \cdot \vec{b}_\mu$ and

$$\boxed{V(\phi^\dagger \phi) = \mu^2 (\phi^\dagger \phi) + |\lambda| \, (\phi^\dagger \phi)^2}$$

▷ Add a Yukawa interaction

$$\mathcal{L}_{\mathrm{Yukawa}} = -\zeta_e \left[\overline{\mathsf{R}}(\phi^\dagger \mathsf{L}) + (\overline{\mathsf{L}}\phi)\mathsf{R} \right]$$

▷ Arrange self-interactions so vacuum corresponds to a broken-symmetry solution: $\mu^2 < 0$
Choose minimum energy (vacuum) state for vacuum expectation value

$$\langle\phi\rangle_0 = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}, \quad v = \sqrt{-\mu^2/|\lambda|}$$

Hides (breaks) $SU(2)_L$ and $U(1)_Y$

but preserves $U(1)_{\rm em}$ invariance

Invariance under $\mathcal{G}$ means $e^{i\alpha\mathcal{G}}\langle\phi\rangle_0 = \langle\phi\rangle_0$, so $\mathcal{G}\langle\phi\rangle_0 = 0$

$$\tau_1\langle\phi\rangle_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} = \begin{pmatrix} v/\sqrt{2} \\ 0 \end{pmatrix} \neq 0 \quad \text{broken!}$$

$$\tau_2\langle\phi\rangle_0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -iv/\sqrt{2} \\ 0 \end{pmatrix} \neq 0 \quad \text{broken!}$$

$$\tau_3\langle\phi\rangle_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ -v/\sqrt{2} \end{pmatrix} \neq 0 \quad \text{broken!}$$

$$Y\langle\phi\rangle_0 = Y_\phi\langle\phi\rangle_0 = +1\langle\phi\rangle_0 = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \neq 0 \quad \text{broken!}$$

Examine electric charge operator Q on the (electrically neutral) vacuum state

$$
\begin{aligned}
Q\langle\phi\rangle_0 &= \tfrac{1}{2}(\tau_3 + Y)\langle\phi\rangle_0 \\
&= \tfrac{1}{2}\begin{pmatrix} Y_\phi + 1 & 0 \\ 0 & Y_\phi - 1 \end{pmatrix}\langle\phi\rangle_0 \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \textit{unbroken!}
\end{aligned}
$$

Four original generators are broken

electric charge is not

▷ $SU(2)_L \otimes U(1)_Y \to \ U(1)_{\text{em}}$ (will verify)

▷ Expect massless photon

▷ Expect gauge bosons corresponding to

$$\tau_1,\ \tau_2,\ \tfrac{1}{2}(\tau_3 - Y) \equiv K$$

to acquire masses

Expand about the vacuum state

Let $\phi = \begin{pmatrix} 0 \\ (v+\eta)/\sqrt{2} \end{pmatrix}$; in *unitary gauge*

$$\begin{aligned} \mathcal{L}_{\text{scalar}} &= \tfrac{1}{2}(\partial^\mu \eta)(\partial_\mu \eta) - \mu^2 \eta^2 \\ &\quad + \frac{v^2}{8}[g^2 \left|b_1 - ib_2\right|^2 + (g'\mathcal{A}_\mu - g b_\mu^3)^2] \\ &\quad + \text{interaction terms} \end{aligned}$$

Higgs boson η has acquired (mass)^2 $M_H^2 = -2\mu^2 > 0$

$$\frac{g^2 v^2}{8}(\left|W_\mu^+\right|^2 + \left|W_\mu^-\right|^2) \Longleftrightarrow M_{W^\pm} = gv/2$$

Now define othogonal combinations

$$Z_\mu = \frac{-g'\mathcal{A}_\mu + g b_\mu^3}{\sqrt{g^2 + g'^2}} \qquad A_\mu = \frac{g\mathcal{A}_\mu + g' b_\mu^3}{\sqrt{g^2 + g'^2}}$$

$$M_{Z^0} = \sqrt{g^2 + g'^2}\, v/2 = M_W \sqrt{1 + g'^2/g^2}$$

A_μ remains massless

$$\mathcal{L}_{\text{Yukawa}} = -\zeta_e \frac{(v+\eta)}{\sqrt{2}}(\bar{e}_{\mathsf{R}} e_{\mathsf{L}} + \bar{e}_{\mathsf{L}} e_{\mathsf{R}})$$
$$= -\frac{\zeta_e v}{\sqrt{2}}\bar{e}e - \frac{\zeta_e \eta}{\sqrt{2}}\bar{e}e$$

electron acquires $m_e = \zeta_e v/\sqrt{2}$

Higgs coupling to electrons: m_e/v ($\propto$ mass)

Desired particle content ... + Higgs scalar

Values of couplings, electroweak scale v?

What about interactions?